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INDUCTIVE PROBABILITY AND THE PARADOX OF IDEAL EVIDENCE*

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Summary

In section I the notions of logical and inductive probability will be discussed as well as two explicanda, viz. degree of confirmation, the base for inductive probability, and degree of evidential support, Popper's favourite explicandum.

In section II it will be argued that Popper's paradox of ideal evidence is no paradox at all; however, it will also be shown that Popper's way out has its own merits.

Abbreviations

lp logical probability
inp inductive probability
dc degree of confirmation
rbq rational betting quotient
des degree of evidential support (or of corroboration)

I. Logical and inductive probability

I.1 Logical probability (lp)

Kemeny has defined[1] a logical measure function, or *logical probability function*, lp on the set of sentences of a language. This measure function is applicable to all functional languages of finite

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order.

If A is a sentence then lp (A) is the ratio of the number of permissible models in which A is true to the total number of permissible models (i.e. models in which the meaning postulates of the language are true). The base of this definition is a rigorous application of the *principle of indifference* to the set of permissible models. This principle, which is fundamental for the *classical* foundation of probability, prescribes to assign the same probability to two 'events' if there is no reason to assume that they are not equally probable (or equipossible)[2]. It is clear that in the definition of lp all permissible models, or more precisely, all sentences describing a permissible model, are regarded as equally probable.

With the logical probability lp we can define the *conditional logical probability* (of the truth) of a sentence A on the basis (of the truth) of a sentence B, lp (A|B), in the usual way as

$$\frac{lp\,(A\&B)}{lp\,(B)}$$

provided that lp (B) ≠ O.

I.2 Degree of evidential support and degree of confirmation.

1) Popper argued in 1954[3] convincingly that his explicandum degree of corroboration or, in my terminology, *degree of evidential support* (des) could not be explicated as a probability function but perhaps as a function of a probability function[4].

Batens' definition[5] seems to me the most interesting definition for des that has been proposed up to now. It can be given explicitly for an uninterpreted probability function p. Des(h|e), the degree of evidential support of h (the hypothesis) by e (the evidense), is :

$$\frac{p(e|h)}{p(e|h)+p(e|\sim h)}$$

Some aspects of this definition of des are important for II.3 :

a) des (h|e) =

$$\frac{1}{1+\dfrac{\dfrac{1}{p(h|e)}-1}{\dfrac{1}{p(h)}-1}}$$

b) des($\sim$h|h)=O (falsification) $\leqslant$ des (h|e) $\leqslant$ 1 (verification) = des (h|h),
c) des (h|e)=1/2 (neutral support) if and only if p(h|e) = p(h),
d) relative to different evidence, des(h|e) increases (decreases) if and only if p(h|e) increases (decreases).

2) Carnap has expicated his explicandum *degree of confirmation* (dc)[6] and one of his expressions for such an explicatum is logical probability, but I prefer to use only his expression *inductive probability* (inp; see I.3) to refer to his explicatum for dc. Carnap has indicated[7] the difference between the explicanda des and dc and he has clarified dc in terms of betting quotients in order to deduce criteria for an unambiguous definition of the explicatum inp. This approach to inductive logic (as Carnap calls the theory of inp) starts in the decision theory and it will roughly be forrulated here[8].

In the first place the (empirical) *credence function* F_{Lx} for person x over language L has to be introduced. Let A be a sentence of L : Suppose x and his opponent bet on A with the following conditions : if A is true x will get the amount s($\sim$A) from the opponent, if A is false x will give the amount r(A) to the opponent. The betting quotient of x on A is then defined as :

$$\frac{r(A)}{r(A)+s(\sim A)}.$$

$F_{Lx}(A)$, the degree of belief of x in A, is defined as the highest betting quotient with which x is willing to bet on A.

Now criteria that are regarded as rational for a better can be formulated in order to get (non-empirical) rational degrees of belief, or *rational betting quotients* (rbq's), i.e. the specific values of a rational credence function. The program of inductive logic for a particular language has been finished if it is possible to formulate a set of acceptable criteria which determine a rational credence function depending only on the syntactic elements of the language and its sentences and perhaps some parameters; this function will then be called the inductive probability function on the set of sentences of that language.

The most important criteria that Carnap has given[9] are
a, the rational credence function must be *regular* (see below),
b, the rational betting quotients must be invariant for certain extensions of the language,
c, if one uses the rational credence function in a betting situation one *learns from experience* (see below),
d, the principle of indifference is operative if the other criteria do not prescribe otherwise.

ad.a. A credence function for L is called regular if for a bet on any subset of contingent sentences of L based on this credence function there is at least one possible combination of truth-values for those sentences that gives net gain to the better. Now Shimony has proved[10] the so-called Ramsey-De Finetti theorem which says that a credence function is regular (or strictly coherent) if and only if it is a regular probability function, i.e. a probability function that assigns the value zero only to analytic falsehoods. Therefore, if regularity is accepted as desideratum for a rational credence function then this theorem makes it possible to define a rational *conditional* credence function (or credebility function as Carnap calls it) in the usual probabilistic way, but then it excludes the possibility to consider genuine universal sentences for they seemed, at least for Carnap, to get the value O for any reasonable probability function on the set of sentences of a language. Carnap accepted the last consequence and excluded theories in his inductive logic for the time being. Hintikka and Kemeny have introduced theories again, but these attempts will not be considered here.

ad.c This criterion (one learns from experience) will be illustrated in I.3. It is always assumed here that a user of the rational credence function takes into account the total evidence he has in the particular betting situation.

I.3. Inductive probability (inp)

For a (first order) monadic predicate language (including, perhaps, some meaning postulates) the set of criteria leads to an inductive probability function, with one parameter; its values differ in general from the lp-values for that language. Theoretically the difference is that different models may get different probabilities because of criterion c.

For our purposes it is sufficient to state the result of the procedure for a particular monadic predicate language and to show that the inp-values differ, in general, from the lp-values. Let T be a (finite or infinite) denumerable set of individuals $t_1, t_2, t_3, \ldots$; these individuals will be called trials. Let there also be α monadic predicates $P_1, P_2, \ldots P_\alpha$. The only meaning postulates are that these predicates are exhaustive and mutually exclusive relative to any trial. This language will be called $L_{\alpha T}$[11].

Let e^k be such a formulation of 'the result' of the first k trials that it gives the information, for every P_i, how many trials, k_i have resulted in P_i. Let h_i be the hypothesis that the next trial will result in P_i.

Now the criteria lead for $L_{\alpha T}$ to the following 'special values' of inp :

(i) $\text{inp}\,(h_i \mid e^k) = \dfrac{k_i + \lambda / \alpha}{k + \lambda}$

λ is a parameter which must be fixed by the rational better on a positive real number.

With these special values the whole structure of inp for $L_{\alpha T}$ is determined by criterion a and the Ramsey-De Finetti theorem.

With respect to regularity the situation is as follows. Because inp is a probability function we can determine the inp-values of universal sentences with formula (i), even if the number of trials is infinite. If e^k_Δ is such that all trials up to the k-th have resulted in P_i then e.g.

$$\text{inp}\,(\forall t\ P_i\ (t)) = \text{inp}\,(h_i)\ \text{inp}\,(h_i \mid e^1_\Delta) \ldots \text{inp}\,(h_i \mid e^k_\Delta) \ldots = \prod_{k=0}^{\infty} \frac{k + \lambda / \alpha}{k + \lambda} = 0$$

However, $\forall t\ P_i(t)$ is not analytically false. Therefore, if one uses inp also for betting on universal sentences one bets *irregular*.

Criterion c. viz. if inp is used in betting situations then one learns from experience, can be illustrated by the following special values ($k > 0$) :

$$\text{inp}\,(h_i \mid e^k_\Delta) = \frac{k + \lambda/\alpha}{k + \lambda} > 1/\alpha = \text{inp}\,(h_i) = \text{lp}\,(h_i \mid e^k) \text{ (for any } e^k).$$

The inequality between the left and the right term also illustrates that the inp-values differ, in general, from the lp-values. If $\lambda = \infty$ we get the same values for inp and lp and it may be stated generally that lp is a limit case of inp. If $\lambda = 0$, (i) may be called *the straight betting rule*, for then the observed relative frequency is used as betting quotient for the next trial. If $\lambda = 0$ and $k = 0$, (i) gives no value, but it may be defined arbitrarily, e.g. $1/\alpha$.

II. The paradox of ideal evidence

II.1 Rational belief

Popper has criticised[12] the 'subjectivists' who try to give a solution of the problem of induction in terms of rational belief, e.g. Keynes and Carnap. In his approach to inductive logic with the credence function Carnap uses, however, only on very rare occasions the term *rational degree of belief* instead of *rational betting quotient*. Popper and Lakatos[13] seem to prefer to criticise *degree of rational belief* which is never used by Carnap as far as I know.

Popper proposes[14] first to use the expression *degree of rationality of a belief* instead of *degree of rational belief* because of the following reasons :

a) the degree of a belief (i.e. in the terminology of Carnap : the value of the credence function for a certain sentence) is a measure of the strength, the intensity, with which one believes something;

b) now the question arises what the intensity, or the degree of belief, is, to which the belief is rationally justifiable and this question is surely independent of any actual intensity;

c) *degree of rational belief* suggests at least a certain dependency on an actual intensity, but *degree of rationality of a belief* does not.

In my opinion Popper's reasons are even more in favour of Carnap's term *rational degree of belief*.

II.2. The paradox and Popper's way out

Popper has construed the paradox of ideal evidence[15] to show that the degree of rational belief cannot be identified with inp. Suppose the trials are tosses with an unknown coin ($\alpha = 2$) and h_i is the hypothesis that the next trial results in P_i and let e^{*2k} be the (ideal) evidence that of the first 2k trials exactly k trials result in P_1 and P_2 resp.Then, according to formula (i) with $\alpha = 2$ and $\lambda > 0$: inp (h_i) = inp $(h_i \mid e^{*2k}) = 1/2$. If degree of rational belief is identified with inp, then this result means '...that our so-called *'degree of rational belief' in the hypothesis*, $[h_i]$, *ought to be completely uneffected by the accumulated evidential knowledge*, $[e^{*2k}]$, ...'[16] And Popper continues : 'I do not think that this paradox can be solved within the framework of the subjective theory, ...'[17]

According to Popper the only way out is to drop h_i as the relevant hypothesis and to look to the higher-level objective probabilistic hypothesis, h(1/2) : 'the objective probability that a toss with this coin results in P_i is 1/2'. The degree of corroboration (i.e. Popper's term for des) of h(1/2) by e^{*2k} can then be regarded as a measure of the rationality of accepting tentatively, a problematic guess, viz. h(1/2).[18] Popper shows[19] that one of his definitions of des has the property to increase (up to 1) in case of ideal evidence if the number of tosses increases, so that there is no paradoxical aspect here. However, Batens has criticised the method of this proof.[20]

II.3 Is there a paradox in the 'subjective' approach ?

In my opinion there is no paradox at all if we abandon the expression *degree of rational belief* and speak only in terms of rational betting quotients (or rational degrees of belief). But Popper's transition from h_i to h(1/2) gives the possibility to disentangle the main intuitions which play a part in case of ideal evidence. The following questions may serve for this purpose :
a) what is the rbq on h_i before and after ideal evidence ?
b) is h_i positively, negatively, or neutrally supported by the observed facts ?
c) what is the rbq on h(1/2) before and after ideal evidence ?
d) is h(1/2) positively, negatively, or neutrally supported by the observed facts ?

Now ideal evidence seems to have only paradoxical results for rbq if one confuses rbq and des as well as questions b and d, for the intuitive answers to the different questions seem to be :
a') the rbq on h_i is before and after ideal evidence the same, viz. 1/2
b') ideal evidence gives neutral support to h_i,
c') the rbq on h(1/2) increases, in case of ideal evidence,proportional to the number of trials,
d') ideal evidence supports h(1/2) positively, and this support increases proportional to the number of trials.

These answers are in perfect agreement with Carnap's approach and the definition of des by Batens, as may be illustrated by the following considerations.
ad a') If inp is regarded as explicatum of rbq, then according to formula (i) the rbq's on h_i before and after ideal evidence do not differ (from 1/2).
ad b') If inp is used as interpretation of p in Batens' definition of des (I.2), we get des $(h_i \mid e{*}2k) = 1/2$ (for any k), which is the value for neutral support.
ad c') If the number of trials is finite (2N) then h(1/2) is the disjunction of all those conjunctive sentences which contain N conjuncts of the form P_1 (t_i) and N conjuncts of the form P_2 (t_i). In this case it can be proved that, if $k < l \leqslant N$, then inp $(h(1/2) \mid e{*}2k) <$ inp $(h(1/2) \mid e{*}2l) \leqslant$ inp $(h(1/2) \mid e{*}2N) = 1$, which is equivalent to the desired property. As far as I know however, one has not succeeded yet in constructing an inductive probability function for infinite languages containing hypotheses like h(1/2) (in general, h(x), $0 \leqslant x \leqslant 1$). But the impossibility of such a construction has not been proved either. The program seems to have a reasonable chance if, in the light of the Ramsey-De Finetti theorem and the universal

character of h(x), one considers several limit procedures and different formulations of the evidence and restrictions on the possible values of x. If this program succeeds the constructed function will certainly satisfy the intuitive requirement formulated in c'.

ad d') If the preceding problem is soluted in a satisfactory way, our intuitions about question d are automatically fulfilled by the definition of des, which is easy to verify.

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NOTES

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[1] J. K. Kemeny, 'A logical Measure Function', *The Journal of Symbolic Logic*, 18 (1953), 289-308. Kemeny's measure function is a generalization of the m^+-function which Carnap has defined in his *Logical Foundations of Probability*, 1950.

[2] The problem of the foundation of probability is now replaced by the problem of which interpretations of the axioms of the probability calculus are useful for some purpose. In some interpretations the principle of indifference is rigorously applied, in others it is applied in a restricted way.

[3] The argument is reprinted in : K. Popper, *The Logic of Scientific Discovery*, 1959, appendix *ix, 387-395.

[4] Some such definitions are discussed in K. Popper, o.c., appendix *ix, 395-406.

[5] D. Batens, 'Some proposals for the solution of the Carnap-Popper discussion on 'inductive logic'', *Studia Philosophica Gandensia*, 6 (1968), 17-18.
Batens uses 'confirmation', where I use 'evidential support'.

[6] E.g. R. Carnap, *Logical Foundations of Probability*, 1950, and *The Continuum of Inductive Methods*, 1952.

[7] See his preface to the second edition of his *Logical Foundations of Probability*, 1962.

[8] For details : R. Carnap, 'The Aim of Inductive Logic', *Logic, Methodology and the Philosophy of Science*, I, 1962, 303-318.

[9] See footnote 8; see also : R. Carnap, 'An Axiom System for Inductive Logic', in P. A. Schilpp (ed.), *The Philosophy of Rudolf Carnap*, 1963, III-V-26.

[10] Shimony, 'Coherence and the axioms of confirmation', *The Journal of Symbolic Logic*, 20 (1955).

[11] Kemeny has chosen $L_{\alpha T}$ in his 'Carnap's Theory of Probability and Induction', P. Schilpp, o.c., II-22.
The particular language discussed by Carnap in his *The Continuum of Inductive Methods*, o.c., differs only from $L_{\alpha T}$ in that it does not have any meaning postulate. The relevant inductive probability function can easily be obtained from that of $L_{\alpha T}$.

[12] E.g. K. Popper, o.c., appendix *ix, 406-419.

[13] I. Lakatos, 'Changes in the Problem of Inductive Logic', in I. Lakatos (ed.), *The Problem of Inductive Logic*, 1968, especially 357-359.

[14] K. Popper, o.c., 407.

[15] Ibid., 407-408.

[16] Ibid., 408.

[17] Ibid., 408.

[18] Ibid., 415.

[19] Ibid., 410-412.

[20] D. Batens, o.c., 8-10.